JOURNAL OF APPROXIMATION THEORY **86**, 144–178 (1996)
ARTICLE NO. 0060

Polynomial Approximation on Varying Sets

NATALY CHISTYAKOVA

Mathematics Department, Kharkov State University, 4 Independence Square, Kharkov 31077, Ukraine

AND

VLADIMIR LOGVINENKO

138 Genoa Street, Arcadia, California 91006
E-mail: FNSM41@prodigy.com

Communicated by Hans Wallin

Received March 16, 1994; accepted in revised form July 25, 1995

Let $\{\gamma_m\}_{m=1}^{\infty}$ be a sequence of positive numbers, and let $f\colon \mathbf{R}^d \to \mathbf{C}$ be a function such that for some $C = C_f < \infty$ and every $\xi \in \mathbf{R}^d$ there exist polynomials $P_m(x) = P_m(x; \xi)$, $\deg P_m \leqslant m$, $m = 0, 1, ...$, satisfying inequalities

$$\sup\{|f(x) - P_m(x; \xi)| : |x - \xi| \leqslant \gamma_m\} \leqslant C \exp\{-m\}.$$

In this paper the authors study smoothness, quasianalytic and analytic properties of f in terms of the sequence $\{\gamma_m\}_{m=1}^{\infty}$. The results are new even for the case that P_m are Taylor polynomials. Using them, the authors prove a Cartwright-type theorem on entire functions of exponential type bounded on some discrete subset of the real hyperplane and construct such a weight-function $\varphi\colon \mathbf{R}^d \to \mathbf{R}$, $d > 1$, that algebraic polynomials are dense in $C^0_{\varphi|A}(A)$ for every affine subspace $A \subset \mathbf{R}^d$ of dimension less than d, but are not dense in the space $C^0_{\varphi}(\mathbf{R}^d)$. © 1996 Academic Press, Inc.

INTRODUCTION

In what follows we use the standard notations of multidimensional analysis.

Let a function $f\colon \mathbf{R}^d \to \mathbf{C}$ have all derivatives up to the order m at the point ξ. We denote by $T_m(x; \xi)$ its Taylor polynomial

$$\sum_{|k|_1 \leqslant m} D^k f(\xi)(x - \xi)^k / k!$$

0021-9045/96 $18.00

centered at the point ξ. Here, $k=(k_1, ..., k_d)\in(\mathbf{Z}_+)^d$ is a multiindex; $|k|_1=k_1+\cdots+k_d$; $k_1!\cdots k_d!$; $z^k=z_1^{k_1}\cdots z_d^{k_d}$ for every $z=(z_1, ..., z_d)\in\mathbf{C}^d$;

$$D^k f(\xi)=\frac{\partial^{|k|_1}f}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(\xi).$$

We denote various constants by C.

Let $\{\gamma_m\}_{m=1}^{\infty}$ be a sequence of positive numbers, and let $P\{\gamma_m\}$ be a linear space of all functions $f\colon \mathbf{R}^d\to\mathbf{C}$ such that

$$\exists C=C_f\,\exists m_0=m_0(f)\in\mathbf{Z}_+(\forall m\in\mathbf{Z}_+ : m\geqslant m_0)\;\forall\xi\in\mathbf{R}^d$$

$$\left(\exists\ \textit{polynomial}\ P_m(x)=P_m(x;\xi)=\sum_k b_k(\xi)\,x^k :\right.$$

$$\left.\deg P_m(x)=\max\{|k|_1 : |b_k|\neq 0\}\leqslant m\right):$$

$$\sup\{|f(x)-P_m(x;\xi)| : x\in\mathbf{R}^d,\ |x-\xi|\leqslant\gamma_m\}\leqslant C\exp\{-m\}.$$

Here, $|z|^2=|z_1|^2+\cdots+|z_d|^2$ for every $z=(z_1, ..., z_d)\in\mathbf{C}^d$.

Let $K(\xi)$ be a positive continuous function defined on $\mathbf{R}^d$. By $P\{\gamma_m; K(\xi)\}$ we denote a linear space of all functions $f\colon \mathbf{R}^d\to\mathbf{C}$ such that

$$\exists C=C_f\,\exists m_0=m_0(f)\in\mathbf{Z}_+(\forall m\in\mathbf{Z}_+ : m\geqslant m_0)\;\forall\xi\in\mathbf{R}^d$$

$$(\exists\ \textit{polynomial}\ P_m(x)=P_m(x;\xi) : \deg P_m(x)\leqslant m):$$

$$\sup\{|f(x)-P_m(x;\xi)| : x\in\mathbf{R}^d,\ |x-\xi|\leqslant\gamma_m\}\leqslant CK(\xi)\exp\{-m\}.$$

Using the same sequence $\{\gamma_m\}_{m=1}^{\infty}$, we define two other linear spaces. We call the first of them $T\{\gamma_m\}$. It consists of all functions $f\in C^{\infty}(\mathbf{R}^d)$ such that

$$\begin{gathered}\exists C=C_f\,\exists m_0=m_0(f)\in\mathbf{Z}_+(\forall m\in\mathbf{Z}_+ : m\geqslant m_0)\;\forall\xi\in\mathbf{R}^d:\\ \sup\{|f(x)-T_m(x;\xi)| : x\in\mathbf{R}^d,\ |x-\xi|\leqslant\gamma_m\}\leqslant C\exp\{-m\}.\end{gathered}\tag{1}$$

The second space is generated by a function $K(\xi)$, together with the sequence $\{\gamma_m\}_{m=1}^{\infty}$. We denote it by $T\{\gamma_m; K(\xi)\}$. It consists of all functions $f\in C^{\infty}(\mathbf{R}^d)$ such that

$$\exists C=C_f\,\exists m_0=m_0(f)\in\mathbf{Z}_+(\forall m\in\mathbf{Z}_+ : m\geqslant m_0)\;\forall\xi\in\mathbf{R}^d:$$

$$\sup\{|f(x)-T_m(x;\xi)| : x\in\mathbf{R}^d,\ |x-\xi|\leqslant\gamma_m\}\leqslant CK(\xi)\exp\{-m\}.$$

It is evident that $T\{\gamma_m\}\subset P\{\gamma_m\}$ and $T\{\gamma_m; K(\xi)\}\subset P\{\gamma_m; K(\xi)\}$. If $K(\xi)\geqslant C>0$ on $\mathbf{R}^d$, the inclusions $T\{\gamma_m\}\subset T\{\gamma_m; K(\xi)\}$ and $P\{\gamma_m\}\subset P\{\gamma_m; K(\xi)\}$ are also true.

If $\gamma_m \leqslant \beta_m$ for all m, $P\{\beta_m; K(\xi)\}$ is a subspace of $P\{\gamma_m; K(\xi)\}$ and $T\{\beta_m; K(\xi)\}$ is a subspace of $T\{\gamma_m; K(\xi)\}$.

The subject of our research considerably differs from the local approximation theory developed by A. Yu. Brudnyi [Br] because of our rigid connection between the size of sets on which polynomials realize approximation and the degree of approximation.

The paper includes four sections. In the first section we study elementary properties of the introduced spaces and differentiability of their elements. The following theorem describes some elementary properties of these spaces:

THEOREM 1. *The following three statements are true*:

(i) *For any* $K(\xi)$ *the inclusion* $P\{\gamma_m; K(\xi)\} \subset C\{\mathbf{R}^d\}$ *is valid*;

(ii) *If* $f \in P\{\gamma_m\}$, *it cannot grow along* $\mathbf{R}^d$ *faster than some exponential* $C \exp\{C\,|x|\}$.

(iii) *Let* $f \in T\{\gamma_m\}$, *and let* m_0 *be the number from* (1). *Then all partial derivatives of* f *of order* m_0 *or more are bounded on* $\mathbf{R}^d$. *Therefore*, f *cannot grow along* $\mathbf{R}^d$ *faster than some polynomial of degree* m_0.

Theorems 2 and 3 describe the relationship between the introduced above spaces and $C^n(\mathbf{R}^d)$.

THEOREM 2. (i) *If the inequality*

$$\liminf_{m \to \infty} (\log(1/\gamma_m))/m > 1/n \tag{2}$$

holds for some $n \in \mathbf{N}$, *then*

$$\{f \in C^n(\mathbf{R}^d) : ((\forall k \in (Z_+)^d : |k|_1 = n) : \sup\{|D^k f(x)| : x \in \mathbf{R}^d\} < \infty)\} \subset P\{\gamma_m\};$$

(ii) *Let* (2) *hold, and let* f *be an arbitrary element of* $C^n(\mathbf{R}^d)$. *There exists such a positive continuous function* $K(\xi)$ *that* $f \in P\{\gamma_m; K(\xi)\}$.

The following statement is an evident consequence of Theorem 2:

COROLLARY 1. *If*

$$\liminf_{m \to \infty} (\log(1/\gamma_m))/m > 0, \tag{3}$$

then every function $f \in C^\infty(\mathbf{R}^d)$ *belongs to* $P\{\gamma_m\}$ *provided that all its derivatives of order* $m \geqslant m_0(f)$, $m_0(f) < \infty$, *are bounded.*

THEOREM 3. *Let* $\limsup_{m\to\infty}(\log(1/\gamma_m))/m < 1/n$ *be valid for some natural n. Then* $P\{\gamma_m; K(\xi)\} \subset C^{n/(2+\varepsilon)}(\mathbf{R}^d)$ *for any positive continuous function* $K(\xi)$ *and any* $\varepsilon > 0$.

It is useful to recall that if $\alpha \notin \mathbf{N}$, $\alpha > 0$, then $C^{\alpha}(\mathbf{R}^d)$ consists of all functions $f \in C^{[\alpha]}(\mathbf{R}^d)$ whose derivatives of the maximal order $[\alpha]$ are elements of the space $Lip_{\alpha-[\alpha]}(\mathbf{R}^d)$. The following statement is a simple consequence of Theorem 3:

COROLLARY 2. *If* $\limsup_{m\to\infty}(\log(1/\gamma_m))/m \leqslant 0$ *then* $P\{\gamma_m; K(\xi)\} \subset C^{\infty}(\mathbf{R}^d)$ *for any positive continuous function* $K(\xi)$.

In the second section we study quasianalyticity of spaces $T\{\gamma_m\}$ and $P\{\gamma_m\}$. To formulate the obtained results, we should recall some definitions.

Let L be a linear subspace of $C^{\infty}(\mathbf{R}^d)$. We say that L is Δ-quasianalytic if the implication

$$(f \in L) \wedge (\exists \xi \in \mathbf{R}^d \; \forall k \in (\mathbf{Z}_+)^d : D^k f(\xi) = 0) \Rightarrow f(x) \equiv 0.$$

is true.

Given a sequence of positive numbers $\{M_\nu\}_{\nu=0}^{\infty}$, $M_0 = 1$, denote by $\{M_\nu^{\sim}\}_{\nu=0}^{\infty}$ a convex regularization of $\{M_\nu\}_{\nu=0}^{\infty}$, i.e. the numbers $\log M_\nu^{\sim}$ are obtained from $\log M_\nu$ by means of the Newton polygonal regularization.

THEOREM 4. *Let* $0 < C_1 \leqslant \gamma_{m-1}/\gamma_m \leqslant C_2 < \infty$. *A space* $T\{\gamma_m\}$ *is* Δ*-quasianalytic if, and only if,*

$$\sum_{m=1}^{\infty} 1/(m/\gamma_m)^* = \infty. \tag{4}$$

Here, $((m/\gamma_m)^*)^m = ((m/\gamma_m)^m)^{\sim}$.

It turns out that the fulfillment of (4) is a criterion of another kind of quasianalyticity for spaces $T\{\gamma_m\}$, and even for spaces $T\{\gamma_m; K(\xi)\}$:

Let $K(\xi)$ *be a continuous function on* $\mathbf{R}^d$. *There is no nontrivial function with a compact support in* $T\{\gamma_m; K(\xi)\}$ *if, and only if, condition* (4) *is fulfilled.*

Of course, we assumed again that

$$0 < C_1 \leqslant \gamma_{m-1}/\gamma_m \leqslant C_2 < \infty.$$

This coincidence is rather striking in comparison with the case of spaces $C\{M_\nu\}(\mathbf{R}^d)$, $d > 1$, where the criteria of these kinds of quasianalyticity are different (later we will recall the definition of $C\{M_v\}(\mathbf{R}^d)$ and the corresponding criteria).

A condition on γ_m close to (4) is necessary and sufficient for the Beurling-type quasianalyticity of spaces $P\{\gamma_m\}$.

THEOREM 5. *Let* $\lim_{m\to\infty} \gamma_m = 0$. *There is no nontrivial function in* $P\{\gamma_m\}$ *vanishing on some set of positive Lebesgue measure if, and only if,*

$$\sum_{m=1}^{\infty} \gamma_m/m = \infty.$$

If $\limsup_{m\to\infty} \gamma_m > 0$, we can say more:

THEOREM 6. *If* $\limsup_{m\to\infty} \gamma_m > 0$, *then for every positive continuous function* $K(\xi)$ *there is not a nontrivial function in* $P\{\gamma_m; K(\xi)\}$ *vanishing on some set of positive Lebesgue measure.*

In the third section we consider spaces whose elements are analytic functions. In view of Theorem 1, it is natural to study spaces $T\{\gamma_m; K(\xi)\}$ and $P\{\gamma_m; K(\xi)\}$ where restrictions on the growth of elements are weaker. The first result of this section is a theorem on holomorphic functions in a layer.

THEOREM 7. *The following three statements are equivalent*:

(i) *The function* f *can be analytically extended from* $\mathbf{R}^d$ *into some layer symmetric with respect to* $\mathbf{R}^d$, *say the layer*

$$\{z = x + iy \in \mathbf{C}^d : x,\ y \in \mathbf{R}^d,\ |y| < H\}, \qquad H > 0;$$

(ii) *There exist a positive sequence* $\{\gamma_m\}_{m=1}^{\infty}$ *and a positive continuous function* $K(\xi)$ *such that*

$$\liminf_{m\to\infty} \gamma_m > 0$$

and $f \in T\{\gamma_m; K(\xi)\}$;

(iii) *There exist a positive sequence* $\{\gamma_m\}_{m=1}^{\infty}$ *satisfying the previous condition and a positive continuous function* $K(\xi)$ *such that* $f \in P\{\gamma_m; K(\xi)\}$.

Let $f: \mathbf{C}^d \to \mathbf{C}$ be an entire function. We say that it is of order ρ if

$$\limsup_{|z|\to\infty} (\log\log |f(z)|)/\log |z| = \rho.$$

Let $\rho \in (0, \infty)$, and let f be an entire function of order ρ. We say that f is of type σ with respect to the order ρ if

$$\limsup_{|z| \to \infty} (\log |f(z)|)/|z|^\rho = \sigma.$$

If $\rho = 1$ and $\sigma \in (0, \infty)$, f is called an entire function of exponential type. We denote by $[\rho; \sigma]_d$ the set of all entire functions in $\mathbf{C}^d$ of type σ or less with respect to the order ρ. Also denote

$$[\rho; \infty)_d = \bigcup_{\sigma > 0} [\rho; \sigma]_d.$$

THEOREM 8. *The following five statements are equivalent*:

(i) *A function f can be extended from $\mathbf{R}^d$ into $\mathbf{C}^d$ as an element of $[\rho; \infty)_d$*;

(ii) *For some $p > 0$ and $A < \infty$ $f \in T\{pm^{1/\rho}; \exp\{A\,|\xi|^\rho\}\}$*;

(iii) *For some $q > 0$ and $B < \infty$ $f \in P\{qm^{1/\rho}; \exp\{B\,|\xi|^\rho\}\}$*;

(iv) *There exist $p > 0$ and $\xi \in \mathbf{R}^d$ such that $f \in C^\infty(\{\xi\})$ and*

$$\exists C\, \forall m \in \mathbf{N} : \sup\{|f(x) - T_m(x; \xi)| : |x - \xi| \leqslant pm^{1/\rho}\} \leqslant C \exp\{-m\};$$

(v) *There exist $q > 0$, $\xi \in \mathbf{R}^d$, and polynomials $P_m(x) = P_m(x; \xi)$, $\deg P_m \leqslant m$, $m \in \mathbf{N}$, such that*

$$\exists C\, \forall m \in \mathbf{N} : \sup\{|f(x) - P_m(x; \xi)| : |x - \xi| \leqslant qm^{1/\rho}\} \leqslant C \exp\{-m\}.$$

We say that $f \in C^\infty(\{\xi\})$ if there exist such neighborhoods U_j, $j \in \mathbf{Z}_+$, of ξ that

$$U_1 \supset U_2 \supset \cdots; \ f \in C^j(U_j), \qquad j \in \mathbf{N}.$$

In this context increasing of the smoothness corresponds to decreasing of the difference between Taylor polynomials and polynomials of the best approximation. For example, this difference is irrelevant at the level of the order of entire functions.

The last, fourth, section is devoted to applications of the developed theory. To formulate the main result of this section, we need the following definition:

A set $E \subset \mathbf{R}^d$ is called an ε-net if

$$\forall x \in \mathbf{R}^d\, \exists \xi \in E : |x - \xi| < \varepsilon.$$

THEOREM 9. *Let positive numbers ε and σ satisfy the condition*

$$200\varepsilon\sigma < \pi. \tag{5}$$

Then the inequality

$$\sup\{|f(x)| : x \in \mathbf{R}^d\} \leqslant \sup\{|f(\xi)| : \xi \in E\}/(1-\varepsilon\sigma)$$

holds for every ε*-net* E *in* $\mathbf{R}^d$ *and every entire function* $f \in [1;\sigma]_d$.

Such sets E that for some constant C the estimate

$$\sup\{|f(x)| : x \in \mathbf{R}^d\} \leqslant C \sup\{|f(\xi)| : \xi \in E\}$$

holds for all functions $f \in [1;\sigma]_d$ are usually called normalizing (for given σ). However, we prefer to call them Cartwright sets in honor of M. L. Cartwright who proved the first result in this area [C]:

For every $\sigma \in (0, \pi)$ *there exists such a finite value* C_σ *that the inequality*

$$\sup\{|f(x)| : x \in \mathbf{R}\} \leqslant C_\sigma \sup\{|f(m)| : m \in \mathbf{Z}\}$$

is valid for all functions $f \in [1;\sigma]_1$.

Cartwright's result was generalized and refined quite a few times (see, for example, [Bo], [BS], [Ber], [Ak], [Le1], [Le2], [A], [M], and [DL]; in the last the complete description of all one-dimensional Cartwright sets is given), but all these authors studied only functions of one variable. It is easy to see that the direct product of d one-dimensional Cartwright sets for given σ is a Cartwright subset of $\mathbf{R}^d$ for this σ. The first result on general but massive (in the sense of Lebesgue measure) Cartwright subsets of $\mathbf{R}^d$ was proved by B. Ya. Levin in 1971 [Le3]. Discrete Cartwright subsets of $\mathbf{R}^d$ were studied in [Lo1], [Lo2] where the theorems similar to Theorem 9 were proved. The essential difference between these theorems and Theorem 9 is the independence of inequality (5) from dimension. Using more complicated calculation, we can reduce 200 to 16 in (5). Of course, the constant 16 is far from the best—for $d=1$ the best constant in (5) is 1.

Cartwright-type theorems is a powerful tool for an analyst because of the ability to "improve" estimates extending them from subsets of $\mathbf{R}^d$ onto this space. Mention only their applications to the probability theory [Lo3], edge-of-the-wedge theorems [Lo4], and the theory of Radon transform [LS]. Here, we will use Theorem 9 for the classical problem of weighted approximation by polynomials on $\mathbf{R}^d$. Let us recall some definitions.

Let $\varphi: \mathbf{R}^d \to \mathbf{R}_+$ be such a function that $\varphi \geqslant 1$ and

$$\forall m \in \mathbf{N}: \lim_{|x| \to \infty} |x|^m/\varphi(x) = 0.$$

We say that $\varphi(x)$ is a weight and define the corresponding weighted space

$$C^0_\varphi = C^0_\varphi(\mathbf{R}^d) = \{f \in C(\mathbf{R}^d) : \lim_{|x| \to \infty} f(x)/\varphi(x) = 0\}.$$

If we set $\|f\| = \sup\{|f(x)|/\varphi(x) : x \in \mathbf{R}^d\}$, C^0_φ is a normed space. If, besides, $\varphi(x)$ is bounded on each compact, C^0_φ appears to be a Banach space. For $d=1$, there are several criteria of completeness of algebraic polynomials in this space (see [Me], [AkB], and [K]).

It is easy to see that if algebraic polynomials of d variables are dense in the space $C^0_\varphi(\mathbf{R}^d)$, then algebraic polynomials are dense in the space $C^0_{\varphi|A}(A)$ for any affine subspace A of $\mathbf{R}^d$. Using Theorem 9, we prove that this necessary condition is not sufficient.

THEOREM 10. *There exists a weight* $\varphi: \mathbf{R}^d \to \mathbf{R}$, $d > 1$, *with the following properties*:

(i) φ *is bounded on each compact set*;

(ii) *All points of discontinuity of* φ *are removable, and the set of all these points is uniformly discrete*;

(iii) *For any affine subspace* $A \subset \mathbf{R}^d$, $\dim A < d$, *algebraic polynomials are dense in the space* $C^0_{\varphi|A}(A)$;

(iv) *Algebraic polynomials are not dense in the space* $C^0_\varphi(\mathbf{R}^d)$.

ELEMENTARY PROPERTIES OF SPACES $P\{\gamma_m\}$, $P\{\gamma_m; K(\xi)\}$, $T\{\gamma_m\}$, $T\{\gamma_m; K(\xi)\}$ AND DIFFERENTIABILITY OF THEIR ELEMENTS (PROOFS OF THEOREMS 1, 3, AND 3)

Proof of Theorem 1. (i) Let ξ be an arbitrary point of $\mathbf{R}^d$. Then for every sufficiently large $m \in \mathbf{N}$

$$\limsup_{x \to \xi} |f(x) - f(\xi)| \leqslant \lim_{x \to \xi} |P_m(x; \xi) - P_m(\xi; \xi)| + CK(\xi) \exp\{-m\} = CK(\xi) \exp\{-m\}.$$

Since m is arbitrary, f is continuous at the point ξ.

(ii) Let $f \in P\{\gamma_m\}$, and let C and m_0 be the constants from the definition of this space. Let $K = C \exp\{-m_0\}$, and let

$$\xi^{(k)} = (\xi^{(k)}_1, ..., \xi^{(k)}_d) = (\gamma_{m_0}/\sqrt{d})\, k \in \mathbf{R}^d,$$
$$Q_k = \{x = (x_1, ..., x_d) \in \mathbf{R}^d : |x_j - \xi^{(k)}_j| \leqslant \gamma_{m_0}/\sqrt{d},\ j = 1, ..., d\},$$
$$\hat{P}_k(x) = P_{m_0}(x; \xi^{(k)})$$

for every $k \in \mathbf{Z}^d$. We call the two cubes Q_k and Q_l neighboring if

$$\max\{|k_j - l_j| : j = 1, ..., d\} = 1.$$

The corresponding polynomials $\hat{P}_k$ and $\hat{P}_l$ are also called neighboring. It is evident that the intersection of two neighboring cubes contains at least one cube with edges parallel to the coordinate axes and the sidelength equal to $\gamma_{m_0}/\sqrt{d}$. Every cube Q_k, $k=(k_1, ..., k_d)$, can be connected with Q_0 by the chain of neighboring cubes. The least length of this chain is $L=\max\{|k_j| : j=1, ..., d\}$. Denote by Q_0, $Q_{k^{(1)}}$, ..., $Q_{k^{(L)}}=Q_k$ the cubes of this minimal chain. The inequality

$$|\hat{P}_{k^{(1)}}(x)-\hat{P}_0(x)| \leqslant 2K$$

holds at every point x of the intersection $Q_{k^{(1)}} \cap Q_0$. By the S. Bernstein Inequality the estimate

$$|\hat{P}_{k^{(1)}}(x)| \leqslant \{2K+\alpha\}(C/\gamma_{m_0})^{m_0}$$

where $\alpha=\max\{|\hat{P}_0(x)| : x \in Q_0\}$ is valid at every point $x \in Q_{k^{(1)}}$. Repeating this argument, we obtain that

$$|\hat{P}_{k^{(2)}}(x)| \leqslant 2K\{(C/\gamma_{m_0})^{m_0}+(C/\gamma_{m_0})^{2m_0}\}+\alpha(C/\gamma_{m_0})^{2m_0}, \qquad x \in Q_{k^{(2)}},$$

$$\cdots$$

$$\begin{aligned}|\hat{P}_k(x)| &= |\hat{P}_{k^{(L)}}(x)| \\ &\leqslant 2K\{(C/\gamma_{m_0})^{m_0}+\cdots+(C/\gamma_{m_0})^{Lm_0}\}+\alpha(C/\gamma_{m_0})^{Lm_0}, \qquad x \in Q_k.\end{aligned}$$

The last one of these estimates implies (ii) because

$$\max\{|f(x)-\hat{P}_k(x)| : x \in Q_k\} \leqslant K.$$

(iii) Let $m \geqslant m_0$ be an arbitrary natural number, and let $\delta=\delta_m=\min\{\gamma_{m-1}, \gamma_m\}$. For every $\xi \in \mathbf{R}^d$ we have

$$\max\left\{\left|\sum_{|k|_1=m} D^k f(\xi)(x-\xi)^k/k!\right| : |x-\xi| \leqslant \delta\right\} \leqslant C(e^{-m+1}+e^{-m}).$$

From this inequality we will deduce estimates for the corresponding partial derivatives using the homogeneity of the differential.

Denote $D^k f(\xi)/k!$ by b_k. The estimate

$$\left|\sum_{|k|_1=m} b_k x^k\right| \leqslant C\,|x|^m$$

holds for all $x \in \mathbf{R}^d$. If $x=(x_1, ..., x_d)$, $y=(y_1, ..., y_d)$, $y_1 \cdots y_d \neq 0$, are arbitrary vectors of $\mathbf{R}^d$, then the inequality

$$\log \left| \sum_{|k|_1 = m} b_k (x+iy)^k \right|$$
$$\leqslant \frac{|y_1 \cdots y_d|}{\pi^d} \int_{\mathbf{R}^d} \left(\log \left| \sum_{|k|_1 = m} b_k t^k \right| \right) \prod_{j=1}^{d} \frac{dt_j}{(x_j - t_j)^2 + y_j^2}$$
$$\leqslant \log C + m \frac{|y_1 \cdots y_d|}{\pi^d} \int_{\mathbf{R}^d} (\log |t|) \prod_{j=1}^{d} \frac{dt_j}{(x_j - t_j)^2 + y_j^2}$$

is valid. In particular, for each $z \in \{z = (z_1, ..., z_d) \in \mathbf{C}^d : |z_j| \leqslant 1,\ j = 1, ..., d\}$ we have

$$\left| \sum_{|k|_1 = m} b_k z^k \right| \leqslant C.$$

By virtue of the Cauchy Inequalities the values of $|b_k|$, $|k|_1 = m$, are bounded by the same constant. Theorem 1 is proved.

Remark 1. The polynomial growth of a function guaranteed by the statement (iii) of Theorem 1 can be obtained under the weaker assumption. The following statement is true:

If $f \in C^m(\mathbf{R}^d)$ is such a function that

$$\exists C < \infty\ \exists \delta > 0\ \forall \xi \in \mathbf{R}^d : \max\{ |f(x) - T_m(x; \xi)| : |x - \xi| \leqslant \delta \} \leqslant C, \quad (6)$$

then f is majorized by some polynomial of degree m.

Proof of Remark 1. We start from the case $d = 1$. Let $f \in C^m(\mathbf{R})$ satisfy estimate (6). The inequalities

$$\left| \int_x^{x+\alpha} f^{(m)}(t)(x + \alpha - t)^{m-1}\, dt \right| \leqslant C,$$
$$\left| \int_{x + \alpha(m-1)/m}^{x+\alpha} f^{(m)}(t)(x + \alpha - t)^{m-1}\, dt \right| \leqslant C$$

hold for every $x \in \mathbf{R}$ and every $\alpha \in [-\gamma, \gamma]$. Here, C does not depend on x and α. Therefore,

$$\left| \int_x^{x + \gamma(m-1)/m} f^{(m)}(t)(x + \gamma - t)^{m-1}\, dt \right| \leqslant 2C = C.$$

Combining this inequality with the inequality

$$\left| \int_x^{x + \gamma(m-1)/m} f^{(m)}(t)((x + \gamma - t) - \gamma/m)^{m-1}\, dt \right| \leqslant C,$$

we obtain that

$$\left| \int_{x}^{x+\gamma(m-1)/m} f^{(m)}(t)((x+\gamma-t)^{m+2}+\cdots)\,dt \right| \leqslant C$$

where the dots stand for all terms of degree less than $(m-2)$. Applying this argument once more, we see that the inequality

$$\left| \int_{x}^{x+\gamma(m-2)/m} f^{(m)}(t)((x+\gamma-t)^{m-3}+\cdots)\,dt \right| \leqslant C$$

holds for all $x \in \mathbf{R}$ and some C that does not depend on x. Continuing this procedure, we obtain that the estimate

$$\left| \int_{x}^{x+\gamma/m} f^{(m)}(t)\,dt \right| \leqslant C$$

is valid for every $x \in \mathbf{R}$. By Newton-Leibnitz Formula we have

$$|f^{(m-1)}(x+\gamma/m)| \leqslant |f^{(m-1)}(x)| + C$$

for every $x \in \mathbf{R}$. It shows that the inequality

$$|f^{(m-1)}(x)| \leqslant C\,|x| + \max\{\,|f^{(m-1)}(\xi)| : -\gamma/m \leqslant \xi \leqslant \gamma/m\}$$

holds for all real x. It is possible only for functions that grow no faster than some polynomial of degree m.

Assume now that $d > 1$ and introduce the family of functions $\varphi_\omega(t) = f(t\omega)$, $t \in \mathbf{R}$, ω is an element of the unit sphere $\mathbf{S}^{d-1}$ of $\mathbf{R}^d$. Our previous argument is valid for every φ_ω. It is easy to see that the constants in the estimates can be chosen independent of ω. Therefore, $f(x)$ cannot grow faster than some polynomial of variable $|x|$ and degree m. Remark 1 is proved.

Proof of Theorem 2. (i) Let $f \in C^n(\mathbf{R}^d)$, and let all its partial derivatives of order n be bounded. Without loss of generality, we can assume that f is real-valued. For all $x \in \mathbf{R}^d$ and all $\xi \in \mathbf{R}^d$ we have

$$|f(x) - T_{n-1}(x;\xi)| = \left| \left\{ (x_1-\xi_1)\frac{\partial}{\partial x_1} + \cdots + (x_d-\xi_d)\frac{\partial}{\partial x_d} \right\}^n f(y)/n! \right|$$

where y is some point of the interval $(x;\xi)$. Therefore,

$$|f(x) - T_{n-1}(x;\xi)| \leqslant C\,|x-\xi|^n$$

where C does not depend on x and ξ. Let $r_m^n = e^{-m}$, i.e., $r_m = e^{-m/n}$. The inequality

$$|f(x) - T_{n-1}(x; \xi)| \leqslant C \exp\{-m\}$$

is valid in the ball $\{x \in \mathbf{R}^d : |x - \xi| \leqslant r_m\}$ for all $\xi \in \mathbf{R}^d$ provided that $m \geqslant n$. Since

$$(\log(1/r_m))/m \equiv 1/n,$$

f belongs to all spaces $P\{\gamma_m\}$ satisfying inequality (2) provided that the first $(n-1)$ γ_m are suitable.

(ii) To prove this statement, we should only change the constant C to the function

$$K(\xi) = \max\left\{\left|\left\{(x_1 - \xi_1)\frac{\partial}{\partial x_1} + \cdots + (x_d - \xi_d)\frac{\partial}{\partial x_d}\right\}^n f(y)/n!\right| : \right.$$

$$\left. |x - \xi| \leqslant 1,\ |y - \xi| \leqslant 1\right\}$$

in the proof of (i). Theorem 2 is proved.

Proof of Theorem 3. We need some facts on the Δ-quasianalyticity. Let $\{M_k\}$, $k \in (\mathbf{Z}_+)^d$, $M_{(0, \ldots, 0)} = 1$, be a sequence of positive numbers, and let

$$\begin{aligned} C\{M_k\} &= C\{M_k\}(\mathbf{R}^d) \\ &= \{f \in C^\infty(\mathbf{R}^d) : \\ &\qquad (\exists L = L_f < \infty\ \forall x \in \mathbf{R}^d\ \forall k \in (\mathbf{Z}_+)^d : |D^k f(x)| \leqslant L^{|k|_1 + 1} M_k)\}. \end{aligned}$$

The criteria of Δ-quasianalyticity of this class are different for $d = 1$ and $d > 1$. If $d = 1$, the following theorem is true:

THEOREM (Denjoy–Carleman–Ostrovsky, [Ma]). *The class $C\{M_k\}(\mathbf{R})$ is Δ-quasianalytic if, and only if,*

$$\sum_{k=1}^{\infty} 1/\sqrt[k]{(M_k)^{\sim}} = \infty.$$

For $M_0 = M_1 = 1$, $M_k = (k \log^2 k)^k$, $k = 2, 3, \ldots$, the corresponding class $C\{M_k\}(\mathbf{R})$ is not Δ-quasianalytic. It is known (see, for example, [Ma]) that it contains a nonnegative function $\omega(t)$ with the following properties:

(i) $\forall t \in \mathbf{R}\, \forall j \in \mathbf{Z}_+ : |\omega^{(j)}(t)| \leqslant M_j$;

(ii) $\exists a \in (0, \infty)$ (say $a = 1$) : $supp\, \omega \subseteq [-a, a]$;

(iii) $\sum_{j=-\infty}^{\infty} \omega(t+j) \equiv 1$.

(i) implies that the estimate

$$\max\left\{\left|\frac{d^j}{dt^j}\,\omega\left(\frac{t-\tau}{h}\right)\right| : t \in \mathbf{R}\right\} \leqslant M_j h^{-j}$$

is valid for every $\tau \in \mathbf{R}$, every $h > 0$, and every $j \in \mathbf{Z}$.

Let $x^{(0)} = (x_1^{(0)}, ..., x_d^{(0)})$ be an arbitrary point of $\mathbf{R}^d$. We will prove that every function $f \in P\{\gamma_m; K(\xi)\}$ is smooth enough on the cube

$$\Pi_{1/2} = \{x = (x_1, ..., x_d) \in \mathbf{R}^d : -1/2 \leqslant x_j - x_j^{(0)} \leqslant 1/2,\ j = 1, ..., d\}.$$

Let $\nu \in \mathbf{N}$ be large enough, and let $N_\nu = 2([\sqrt{d}/\gamma_\nu] + 1)$. Divide

$$\Pi_1 = \{x = (x_1, ..., x_d) \in \mathbf{R}^d : -1 \leqslant x_j - x_j^{(0)} \leqslant 1,\ j = 1, ..., d\}.$$

into N_ν^d equal little cubes $\sigma_p = \sigma_p(\nu)$, $p = 1, ..., N_\nu^d$. Denote by $x^{(p)}$ the center of σ_p and by h_ν its sidelength. It is evident that $\gamma_\nu/(2\sqrt{d}) \leqslant h_\nu \leqslant \gamma_\nu/\sqrt{d}$. By the definition of $P\{\gamma_m; K(\xi)\}$ for every $p \in \{1, ..., N_\nu^d\}$ there exists such a polynomial $P_p(x) = P_\nu(x; x^{(p)})$ that $\deg P_p \leqslant \nu$ and that

$$\max\{|f(x) - P_p(x)| : x \in \sigma_p\} \leqslant C \exp\{-\nu\}, \qquad C = \max\{K(\xi) : \xi \in \Pi_1\}.$$

Therefore, the estimate

$$\left| f(x) - \sum_{p=1}^{N_\nu^d} \left(\prod_{j=1}^{d} \omega((x_j - x_j^{(p)})/h_\nu) \right) P_p(x) \right| \leqslant C \exp\{-\nu\}.$$

holds at every point x of the cube

$$\Pi_{1-h_\nu} = \{x = (x_1, ..., x_d) \in \mathbf{R}^d : -1 + h_\nu \leqslant x_j - x_j^{(0)} \leqslant 1 - h_\nu,\ j = 1, ..., d\}.$$

By Jackson Theorem (see [N, Ch. IV]) for the function $\omega(t/h_\nu)$ and for each $l \in \mathbf{N}$ there exists such a polynomial $Q(t) = Q_l(t)$ that $\deg Q_l \leqslant l$ and

$$\begin{aligned}\max\{|\omega(t/h_\nu) - Q(t)| : -2 \leqslant t \leqslant 2\} &\leqslant C^\nu M_\nu h_\nu^{-\nu} l^{-\nu}\\ &\leqslant (C\nu(\log^2 \nu)/(l\gamma_\nu))^\nu \leqslant (\exp\{\nu/n'\}/l)^\nu.\end{aligned}$$

Here, C is an absolute constant and

$$L = \limsup_{m\to\infty} (\log(1/\gamma_m))/m < 1/n' < 1/n, \qquad n' > 0.$$

If $l = [\exp\{2v/n\}] + 1$, then

$$\max\{|\omega(t/h_v) - Q(t)| : -2 \leqslant t \leqslant 2\} \leqslant \exp\{-v^2/n\}.$$

Define the polynomial

$$R(x) = R_v(x) = \sum_{p=1}^{N_v^d} \left(\prod_{j=1}^{d} Q(x_j - x_j^{(p)}) \right) P_p(x).$$

We have

$$\deg R \leqslant d([\exp\{2v/n\}] + 1) + v.$$

Since

$$\prod_{j=1}^{d} a_j - \prod_{j=1}^{d} b_j = \sum_{i=1}^{d} \left(\prod_{j-1}^{i-1} a_j \right) (a_i - b_i) \left(\prod_{j=i+1}^{d} b_j \right),$$

the estimate

$$\begin{aligned} |f(x) - R(x)| \leqslant & \left| f(x) - \sum_{p=1}^{N_v^d} \left(\prod_{j=1}^{d} \omega((x_j - x_j^{(p)})/h_v) \right) P_p(x) \right| \\ & + \max\{|P_p(x)| : p = 1, ..., N_v^d, x \in \Pi_1\} \\ & \times \sum_{p=1}^{N_v^d} \left| \prod_{j=1}^{d} \omega((x_j - x_j^{(p)})/h_v) - \prod_{j=1}^{d} Q(x_j - x_j^{(p)}) \right| \\ \leqslant & C \exp\{-v\} \\ & + C \max\{|P_p(x)| : p = 1, ..., N_v^d, x \in \Pi_1\} \; N_v^d \exp\{-v^2/n\} \end{aligned}$$

holds at every point

$$x \in \Pi_{2/3} = \{x = (x_1, ..., x_d) \in \mathbf{R}^d : -2/3 \leqslant x_j - x_j^{(0)} \leqslant 2/3, j = 1, ..., d\}.$$

By virtue of S. Bernstein Inequality

$$\max\{|P_p(x)| : p = 1, ..., N_v^d, x \in \Pi_1\} \leqslant (C/\gamma_v)^v$$

where C does not depend on v. Since $N_v = O(\gamma_v^{-1})$, these estimates imply that the inequality

$$\begin{aligned} |f(x) - R(x)| & \leqslant C \exp\{-v\} + (C/\gamma_v)^{d+v} \exp\{-v^2/n\} \\ & \leqslant C \exp\{-v\} + C \exp\{(d+v) v/n'' - v^2/n\} \\ & \leqslant C \exp\{-v\} \end{aligned} \tag{7}$$

holds at every point of $\Pi_{2/3}$ provided that $n'^{-1} < n''^{-1} < n^{-1}$ and ν is large enough. The estimate

$$\log(\deg R_\nu) \leqslant (2+\varepsilon)\,\nu/n \tag{8}$$

is valid for every $\varepsilon > 0$ and all $\nu > \nu_0(\varepsilon)$, $\nu_0(\varepsilon) < \infty$. It follows from (7) and (8) that

$$\max\{|f(x) - R(x)| : x \in \Pi_{1/2}\} \leqslant C/(\deg R_\nu)^{n/(2+\varepsilon)}.$$

To finish the proof of Theorem 3, we should only refer to the well-known S. Bernstein Theorem (see [N, Ch. IV]) that guarantees the inclusion $f \in C^{n/(2+\varepsilon)}$. Theorem 3 is proved.

2. Quasianalyticity (Proofs of Theorems 4, 5, and 6)

Proof of Theorem 4. Sufficiency. Let $f \in T\{\gamma_m\}$. We have the following estimate of its differential:

$$\sup\left\{\left|\sum_{|k|_1=m} D^k f(\xi)(x-\xi)^k/k!\right| : \xi,\, x \in \mathbf{R}^d;\ |x-\xi| \leqslant \min\{\gamma_{m-1}, \gamma_m\}\right\}$$
$$\leqslant C(\exp\{-m+1\} + \exp\{-m\}) \leqslant C\exp\{-m\}.$$

As we have already verified, this estimate implies that the inequality

$$|D^k f(\xi)| \leqslant k!\,\exp\{m\log(1/\gamma_m) + O(m)\} \leqslant \exp\{m\log(m/\gamma_m) + O(m)\} \tag{9}$$

holds for every $\xi \in \mathbf{R}^d$ and all $k \in (\mathbf{Z}_+)^d$, $|k|_1 = m$.

We need the following criterion for Δ-quasianalyticity of classes $C\{M_k\}(\mathbf{R}^d)$, $d > 1$:

Theorem (Matsaev–Ronkin, [MR]). *Class* $C\{M_k\}(\mathbf{R}^d)$, $d>1$, *is Δ-quasianalytic if, and only if, the appropriate classes*

$$C\{M_{(m,0,\ldots,0)}\}(\mathbf{R}), \ldots, C\{M_{(0,\ldots,0,m)}\}(\mathbf{R})$$

are Δ-quasianalytic.

By this theorem we should only verify that the class $C\{M_k\}$ where $M_0 = 1$ and $M_\nu = (\nu/\gamma_\nu)^\nu$, $\nu \in \mathbf{N}$, is Δ-quasianalytic. By virtue of the Denjoy–Carleman–Ostrovsky Theorem the Δ-quasianalyticity is equivalent to the divergence of series (4). The sufficiency is proved.

Necessity. Let

$$\sum_{j=1}^{\infty} 1/(j/\gamma_j)^* < \infty.$$

By the Matsaev–Ronkin Theorem for every $p>0$ there exist such a nontrivial function

$$f = f_p \in C\{k!\gamma_{|k|_1-1}^{-|k|_1}\}$$

and such a point $\xi \in \mathbf{R}^d$ that

$$\forall k \in (\mathbf{Z}_+)^d : D^k f(\xi) = 0;$$

$$\forall k \in (\mathbf{Z}_+)^d : \sup\{|D^k f(x)| : x \in \mathbf{R}^d\} \leqslant k!(p/\gamma_{|k|_1-1})^{|k|_1}, \gamma_{-1} = 1.$$

If p is small enough, the inequality

$$\max\left\{\left|f(x) - \sum_{|k|_1 \leqslant m} D^k f(\xi)(x-\xi)^k/k!\right| : |x-\xi| \leqslant \gamma_m\right\}$$

$$\leqslant \sup\left\{\left|\sum_{|k|_1 = m+1} D^k f(\eta)(x-\xi)^k/k!\right| : x, \xi, \eta \in \mathbf{R}^d; |x-\xi| \leqslant \gamma_m\right\}$$

$$\leqslant \sum_{|k|_1 = m+1} \gamma_m^{-m-1} \gamma_m^{m+1} p^m < (m+1)^d\, p^m \leqslant \exp\{-m\}$$

holds for all $\xi \in \mathbf{R}^d$. Theorem 4 is proved.

Let us prove the statement on another kind of quasianalyticity mentioned in the Introduction. At first, we will verify that there is no nontrivial function with a compact support in the class $T\{\gamma_m\}$. For this purpose, recall the following result:

THEOREM (Lelong, [L], [MR]). *There is no nontrivial function with a compact support in the class* $C\{M_k\}(\mathbf{R}^d)$, $d>1$, *if, and only if, the class* $C\{\hat{M}_m\}(\mathbf{R})$, $\hat{M}_m = \max\{M_k : |k|_1 = m\}$, $m \in \mathbf{Z}_+$, *is* Δ*-quasianalytic.*

While proving the sufficiency in Theorem 4, we really proved that for all $m \in \mathbf{N}$

$$\sup\{|D^k f(\xi)| : |k|_1 = m; \xi \in \mathbf{R}^d\} \leqslant m!\, \exp\{m \log(1/\gamma_m) + O(m)\}.$$

Therefore, the sufficiency of (3) is a consequence of the Lelong Theorem. By the same theorem the function f_p defined in the proof of Theorem 4 can be chosen so that it has a compact support. Since $f_p \in T\{\gamma_m\}$ if $p>0$ is small enough, the necessity of (4) for the quasianalyticity of $T\{\gamma_m\}$ is also proved.

Let $K: \mathbf{R}^d \to \mathbf{R}$ be an arbitrary positive continuous function. It is easy to see that if $f: \mathbf{R}^d \to \mathbf{C}$ has a compact support, then

$$(f \in T\{\gamma_m\}) \Leftrightarrow (f \in T\{\gamma_m; K(\xi)\}).$$

Therefore, our statement is really proved for the general case.

Some concepts and theorems are used in the proofs of both Theorem 5 and Theorem 6. It is convenient to introduce them proving Theorem 6. So, we start with the proof of this theorem.

Proof of Theorem 6. Assume that $d=1$. We prove, to begin with, that $P\{\gamma_m; K(\xi)\}$ is a so-called *I*-quasianalytic space. It means that every function of this space vanishing on some interval vanishes identically. Assume the contrary: There exists such a function $f \in P\{\gamma_m; K(\xi)\}$ that $f_{|[a,b]}=0$, $a<b$, but $f(x)$ does not vanish identically on the union of any left neighborhood of a and any right neighborhood of b. Suppose, for the sake of definiteness, that f does not vanish identically on any left neighborhood of a. Moving b to the left, if necessary, we can assume that

$$\gamma = \limsup_{m \to \infty} \gamma_m > 2(b-a).$$

Let $\{m_\mu\}_{\mu=1}^{\infty}$ be such a subsequence of $\mathbf{N}$ that

$$\gamma = \lim_{\mu \to \infty} \gamma_{m_\mu},$$

and let $R_\mu(x) = P_{m_\mu}(x; (a+b)/2)$, $\mu \in \mathbf{N}$. The sequence of polynomials $R_\mu(x)$, $\mu \in \mathbf{N}$, converges to f uniformly on $[(a+b)/2-\gamma/2, (a+b)/2+\gamma/2]$. Besides, this sequence tends to 0 with exponential degree of convergence on $[a,b]$. By S. Bernstein Inequality this sequence converges to 0 on some interval containing $[a,b]$. It contradicts our assumption. So, $f(x)$ should be 0 identically.

Let us verify now that every function $f \in P\{\gamma_m; K(\xi)\}$ vanishing on some set $e \subset \mathbf{R}$ of the positive Lebesgue measure vanishes on some interval and, therefore, vanishes identically. For this purpose, we need the particular case $d=1$ of the following result:

THEOREM (Schaeffer–Levin). *Let E be a relatively dense subset of $\mathbf{R}^d$, i.e. there exist such constants L and $\delta>0$ (density characteristics of E) that for every $x \in \mathbf{R}^d$*

$$meas_d\, E \cap \{y \in \mathbf{R}^d : |y-x| \leqslant L\} \leqslant \delta.$$

Then the estimate

$$\sup\{|g(x)| : x \in \mathbf{R}^d\} \leqslant \exp\{C\sigma L^{d+1}/\delta\} \sup\{|g(\xi)| : \xi \in E\}$$

holds for every entire function g of exponential type σ. Here $C = C_d$ depends on d only.

This theorem for functions of one variable was proved by A. C. Shaeffer [S] in 1953. The proof of the general case was obtained by B. Ya. Levin [Le3] in 1971. The recent achievements in this area one can find in [LL], [LLS], and [DL].

Without loss of generality we can assume that 0 is a point of density of E. It means that

$$meas(e \cap [-\Delta, \Delta]) = 2\Delta(1-\eta)$$

where $\eta = \eta(\Delta) \to 0$ as $\Delta \to 0$. Define a sequence of entire functions

$$g_\mu(z) = R_\mu(\Delta \cos z), \qquad \mu \in \mathbf{N},$$

of exponential types m_μ respectively. These functions are bounded by constants $C \exp\{-m_\mu\}$, $C = \max\{K(\xi) : -1 \leqslant \xi \leqslant 1\}$, on the relatively dense set

$$E = \{x \in \mathbf{R} : \Delta \cos x \in e\}.$$

Since $\Delta > 0$ is arbitrary, we can assume that the density characteristics of E satisfy the inequality

$$C_1 L^2/\delta < 1/2.$$

Here, C_1 is the coefficient from the Schaeffer-Levin Theorem. By this theorem

$$\max\{|g_\mu(x)| : x \in \mathbf{R}\} \leqslant C \exp\{-m_\mu/2\}, \qquad \mu \in \mathbf{N},$$

or

$$\max\{|P_{m_\mu}(t; 0)| : -\Delta \leqslant t \leqslant \Delta\} \leqslant C \exp\{-m_\mu/2\}, \qquad \mu \in \mathbf{N},$$

So, $f_{|[-\Delta, \Delta]} = 0$, and Theorem 5 is proved for the case where $d = 1$. The general statement can be easily obtained by reduction to the one-variable case. Theorem 6 is proved.

Proof of Theorem 5. Prove, to begin with, Theorem 5 assuming that $d = 1$ and $\gamma_m/m \downarrow 0$ as $m \to \infty$. In this case our argument is similar to Beurling's proof of his well-known Quasianalyticity Theorem [Beu1].

Sufficiency. Let $f \in P\{\gamma_m\}$ vanish on a set e of positive Lebesgue measure. Without loss of generality, we can assume that 0 is a point of density of e. Define

$$F(s) = \int_0^\infty \exp\{-sx\}\, f(x)\, dx.$$

Since by statement (ii) of Theorem 1 the function $f(x)$ does not grow along the real line faster than some exponential, say $C\exp\{-|x|/2\}$, the function $F(s)$ is holomorphic and bounded in the halfplane $\sigma = \operatorname{Re} s \geqslant 1$. Let m_0 be the number from the definition of $P\{\gamma_m\}$. Fix, for a while, a natural number $m \geqslant m_0$, define $x_l = x_l(m) = 3l\gamma_m/2$ and denote by $R_l(x)$ the polynomials $P_m(x; x_l)$ for all $l \in \mathbf{Z}$. There exists a nonnegative function $\omega(x) = \omega_m(x) \in C^\infty(\mathbf{R})$ with the following properties:

(i) $supp\, \omega \subset [-\gamma_m, \gamma_m]$;

(ii) $\omega(x)_{|[-\gamma/2, \gamma/2]} \equiv 1$;

(iii) $|\omega'(x)| \leqslant C/\gamma_m$ for all $x \in \mathbf{R}$;

(iv) $\sum_{l \in \mathbf{Z}} \omega(x - x_l) = 1$.

Extend ω into the complex plane by means of the equality $\omega(z) = \omega(\operatorname{Re} z)$. We have the following representation:

$$F(s) = \int_0^\infty \exp\{-sx\} \left\{ f(x) - \sum_{l \geqslant 0} \omega(x - x_l)\, R_l(x) \right\} dx$$
$$+ \int_0^\infty \exp\{-sx\} \left\{ \sum_{l \geqslant 0} \omega(x - x_l)\, R_l(x) \right\} dx = I_1 + I_2. \qquad (10)$$

We will evaluate I_1 and I_2 separately.

The estimate of I_1 is very simple: If $\sigma = \operatorname{Re} s \geqslant 1$, then

$$|I_1| \leqslant C \exp\{-m\} \int_0^\infty \exp\{-\sigma x\}\, dx \leqslant C \exp\{-m\}. \qquad (11)$$

The estimate of the term I_2 is more complicated. By the Green Formula applied to the rectangle Π_R with the vertices 0, R, $R+i$, i we have

$$\int_{\partial \Pi_R} \exp\{-sz\} \left\{ \sum_{l \geqslant 0} \omega(z - x_l)\, R_l(z) \right\} ds$$
$$= 2i \iint_{\Pi_R} \exp\{-sz\} \left\{ \sum_{l \geqslant 0} \frac{d}{d\bar{z}}\, \omega(z - x_l)\, R_l(z) \right\} dx\, dy, \qquad z = x + iy.$$

Applying the S. Bernstein Inequality to the polynomials R_l and noticing that at every point $z \in \mathbf{C}$ at most two terms of the integrand series are different from 0, we see that the contribution of the right side of Π_R to the integral on the left-hand side tends to 0 as $R \to \infty$. Therefore,

$$\begin{aligned} I_2 = {} & i \int_0^1 \exp\{-isy\}\, R_0(iy)\, dy \\ & + \int_0^\infty \exp\{-s(x+i)\} \sum_{l \geqslant 0} \omega(x+i-x_l)\, R_l(x+i)\, dx \\ & + 2i \iint_\Pi \exp\{-sz\} \left\{ \sum_{l \geqslant 0} R_l(z) \frac{d}{d\bar{z}} \omega(z-x_l) \right\} dx\, dy \\ = {} & I_{21} + I_{22} + I_{23} \end{aligned} \tag{12}$$

where $\Pi = \Pi_\infty$ is the halfstrip $\{z = x+iy \in \mathbf{C} : x > 0,\ 0 < y < 1\}$. Once again, we will evaluate each of the terms separately.

Since $\gamma_m \to 0$ as $m \to \infty$, the Schaeffer-Levin Theorem guarantees that the estimate

$$|R_0(x)| \leqslant \exp\{-m + o(m)\}, \qquad m \to \infty,$$

is valid for all $x \in [-\gamma_m, \gamma_m]$. Therefore, the inequality

$$|R_0(iy)| \leqslant \exp\{-m + o(m)\} (y/\gamma_m + \sqrt{1 + (y/\gamma_m)^2})^m, \qquad m \to \infty,$$

holds for every $y \in [0, 1]$. If $t = \operatorname{Im} s \leqslant -2m/\gamma_m$, then

$$\begin{aligned} |I_{21}| & \leqslant \int_0^1 (y/\gamma_m + \sqrt{1 + (y/\gamma_m)^2})^m \exp\{ty - m + o(m)\}\, dy \\ & \leqslant \int_0^1 (1 + 2y/\gamma_m)^m \exp\{ty - m + o(m)\}\, dy \\ & \leqslant \exp\{-m + o(m)\} \int_0^1 \{(1 + 2y/\gamma_m) \exp\{-2y/\gamma_m\}\}^m\, dy \\ & \leqslant \exp\{-m + o(m)\}, \qquad m \to \infty. \end{aligned}$$

So,

$$|I_{21}| \leqslant \exp\{-m + o(m)\}. \tag{13}$$

To estimate I_{23}, let us notice that the integrand vanishes outside the union of the rectangles

$$\ldots, \{z = x + iy : \gamma_m/2 \leqslant x \leqslant \gamma_m;\ 0 \leqslant y \leqslant 1\},$$
$$\{z = x + iy : 2\gamma_m \leqslant x \leqslant 5\gamma_m/2;\ 0 \leqslant y \leqslant 1\}, \ldots$$

and equals

$$\exp\{-sz\}\ \frac{d}{d\bar{z}}\,\omega(z-x_l)\{R_{l+1}(z)-R_l(z)\}$$

on the rectangle situated between x_l and x_{l+1}. Since

$$|R_{l+1}(z)-R_l(z)|\leqslant 2C\exp\{-m\}$$

on the bottom side of the rectangle, the same argument we used to estimate I_{21} shows that

$$\begin{aligned}|I_{23}|&\leqslant \exp\{-m+o(m)\}\sum_{l\geqslant 0}\exp\{-\sigma x_l/2\}\\&\leqslant \exp\{-m+o(m)\},\qquad m\to\infty.\end{aligned}\tag{14}$$

If $x_l-\gamma_m\leqslant x\leqslant x_l+\gamma_m$ and $0\leqslant y\leqslant 1$, then

$$|R_l(x+iy)|\leqslant \exp\{x/2\}(C/\gamma_m)^m.$$

Consequently, for $t\leqslant -2m/\gamma_m$ we have

$$\begin{aligned}|I_{22}|&\leqslant 2(C/\gamma_m)^m\int_0^{\infty}\exp\{t-x/2\}\,dx\\&\leqslant 4\exp\{m(-\log\gamma_m+\log C-2/\gamma_m)\}<\exp\{-m\}.\end{aligned}\tag{15}$$

(12), (13), (14), and (15) show that the estimate

$$|I_2|\leqslant \exp\{-m/2\}\tag{16}$$

holds for $t=\operatorname{Im}s\leqslant -2m/\gamma_m$ and all natural m that are sufficiently large. From (10), (11), and (16) we deduce that for all $t=\operatorname{Im}s\leqslant -2m/\gamma_m$

$$|F(\sigma+it)|\leqslant 2\exp\{-m/2\}.\tag{17}$$

Without loss of generality, we can assume that $|F(s)|\leqslant 1$ on the line $s=1+it$, $-\infty<t<\infty$. By (17) we have

$$\begin{aligned}&\int_{-\infty}^{\infty}(\log|F(1+it)|/(1+t^2))\,dt\\&\quad\leqslant \frac{1}{2}\sum_{m=m_0}^{\infty}\int_{-2(m+1)/\gamma_{m+1}}^{-2m/\gamma_m}((-m+\log 4)/(1+t^2))\,dt\\&\quad\leqslant -\frac{1}{2}\sum_{m=m_0}^{\infty}m\left(\frac{\gamma_m}{m}-\frac{\gamma_{m+1}}{m+1}\right)+O(1)=-\frac{1}{2}\sum_{m=m_0}^{\infty}\frac{\gamma_m}{m}+O(1)=-\infty.\end{aligned}$$

Therefore, $F(s) \equiv 0$. In the same way one can verify that the left Laplace transform of f vanishes. It means that $f(x) \equiv 0$.

If the sequence $\{\gamma_m/m\}$ is not monotone, we apply such a transposition $\tau\colon \mathbf{N} \to \mathbf{N}$ that $\gamma_{\tau(m)}/\tau(m) \downarrow 0$ as $m \to \infty$ (certainly, we can assume that the values γ_m/m do not repeat) and change m to $\tau(m)$ in the previous argument. For the case $d=1$ the sufficiency is proved.

Necessity. First assume that $d=1$. Let $\gamma_m/m \downarrow 0$. This restriction can be removed by the same argument as in the proof of sufficiency.

We should prove that the convergence of the series $\sum_{m=1}^{\infty} \gamma_m/m$ implies the existence of a nontrivial function $f \in P\{\gamma_m\}$ vanishing on some set of positive Lebesgue measure. Really, we will prove more: There exists a nontrivial function $f \in P\{\hat{\gamma}_m\}$ where $\hat{\gamma}_m = \max\{\gamma_m, 1/(m+1)\}$, $m \in \mathbf{Z}_+$, vanishing on the negative ray. It is evident that $\sum_{m=1}^{\infty} \hat{\gamma}_m/m < \infty$ and $\hat{\gamma}_m/m \downarrow 0$.

Let

$$U(x+iy) = \frac{x}{\pi} \sum_{m=0}^{\infty} \int_{m/(e^3\hat{\gamma}_m) \leqslant |t| \leqslant (m+1)/(e^3\hat{\gamma}_{m+1})} \{m + \log(1+t^2)\} \frac{dt}{x^2+(t-y)^2}.$$

The series of integrals on the right-hand side of this equality converges because

$$\sum_{m=1}^{\infty} \{(m+1)\, \hat{\gamma}_{m+1} - m\hat{\gamma}_m\}/m < \infty.$$

The positive function U is harmonic in the right halfplane. Let $\tilde{U}(z)$ be a harmonic function that is conjugate to U. Define

$$F(z) = \exp\{-U(z) - i\tilde{U}(z)\}.$$

This function is analytic and bounded by 1 in the right halfplane. Besides, it tends to 0 as $z \to \infty$ inside some angles adjacent to the imaginary axes. If

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{-\infty} F(z) \exp\{zt\}\, dz,$$

then f is nontrivial, but vanishes on the negative ray. We will prove that $f \in P\{\hat{\gamma}_m\}$. For this purpose, we define the family of polynomials

$$P_m(t;\tau) = \frac{1}{2\pi i} \int_{-im/(e^3\hat{\gamma}_m)}^{im/(e^3\hat{\gamma}_m)} F(z) \exp\{z\tau\} \sum_{j=0}^{m} \frac{(z(t-\tau))^j}{j!}\, dz, \qquad m \in \mathbf{Z}_+, \quad \tau \in \mathbf{R}.$$

We have

$$
\begin{aligned}
&|f(t)-P_m(t;\tau)|\\
&\leqslant\left|\int_{-i\infty}^{-im/(e^3\hat{\gamma}_m)} F(z)\exp\{z\tau\}\,dz\right|\\
&+\left|\int_{-im/(e^3\hat{\gamma}_m)}^{im/(e^2\hat{\gamma}_m)} F(z)\exp\{z\tau\}\left(\exp\{z(t-\tau)\}-\sum_{j=0}^{m}\frac{(z(t-\tau))^j}{j!}\right)dz\right|\\
&+\left|\int_{im/(e^3\hat{\gamma}_m)}^{i\infty} F(z)\exp\{z\tau\}\,dz\right|=J_1+J_2+J_3. \qquad (18)
\end{aligned}
$$

Again, we evaluate each term separately. The estimate of J_1 and J_3 does not depend on τ.

$$J_1\leqslant\sum_{j=m}^{\infty}\int_{-(j+1)/(e^3\hat{\gamma}_{j+1})}^{-j/(e^3\hat{\gamma}_j)}\exp\{-j\}\,\frac{dy}{1+\gamma^2}<\pi\exp\{-m\}. \qquad (19)$$

The estimate of J_3 is similar to the estimate of J_1:

$$J_3<\pi\exp\{-m\}. \qquad (20)$$

To evaluate J_2, let us assume that $|t-\tau|\leqslant\gamma_m$. Then we have

$$
\begin{aligned}
J_2&\leqslant\frac{2m}{e^3\hat{\gamma}_m}\sum_{j=m+1}^{\infty}\left(\frac{m}{e^3\hat{\gamma}_m}\hat{\gamma}_m\right)^j\Big/ j!\\
&\leqslant\frac{2m}{e^3\hat{\gamma}_m}\sum_{j=m+1}^{\infty}(m/(e^2 j))^j\\
&\leqslant\frac{2m}{e^3\hat{\gamma}_m(1-\exp\{-2\})}\exp\{-2(m+1)\}\leqslant\exp\{-m\}. \qquad (21)
\end{aligned}
$$

(18), (19), (20), and (21) together imply that $f\in P\{\hat{\gamma}_m\}$. For the case $d=1$ the necessity is proved.

Since the restriction of $f\in P\{\gamma_m\}$ on every affine subspace $\{x=(x_1,...,x_d)\in\mathbf{R}^d: x_{i_1}=x^0_{i_1},...,x_{i_l}=x^0_{i_l}\}$ belongs to the space $P\{\gamma_m\}$ of functions of $(d-l)$ variables, the sufficiency can be proved by induction on dimension. To prove the necessity in the general case, we should only let

$$\tilde{f}(x_1,...,x_d)=f(x_1);\ \tilde{P}_m(x_1,...,x_d;\ \xi_1,...,\xi_d)=P_m(x_1;\xi_1)$$

where $f(t)$ and $\{P_m(t;\tau)\}$ were defined in the proof of the necessity in the one-dimensional case. Theorem 5 is proved.

Analyticity (Proofs of Theorems 7 and 8)

Proof of Theorem 7. It is obvious that (ii) implies (iii). Let us verify that (iii) $\Rightarrow$ (i). Fix an arbitrary point $\xi \in \mathbb{R}^d$. The series

$$P_0(x;\xi)+(P_1(x;\xi)-P_0(x;\xi))+\cdots+(P_{m+1}(x;\xi)-P_m(x;\xi))+\cdots$$

converges to f uniformly on the ball $\{x \in \mathbf{R}^d : |x-\xi| \leqslant q/2\}$ where $q = \liminf_{m\to\infty} \gamma_m$. Since the general term of this series tends to 0 exponentially, by the S. Bernstein Inequality its sum is a holomorphic function in some polydisk in $\mathbf{C}^d$ centered at the point ξ. The size of the polydisk does not depend on ξ. Therefore, f can be analytically extended into some layer symmetric with respect to $\mathbf{R}^d$. To finish the proof, we should only verify that (i) $\Rightarrow$ (ii).

Assume that (i) is true. For every $\xi \in \mathbf{R}^d$ the funtion f can be expanded into the power series

$$f(z)=\sum_{k\in(\mathbf{Z}_+)^d} c_k(z-\xi)^k$$

converging uniformly on the polydisk

$$\Pi=\Pi_\xi=\{z=(z_1,...,z_d)\in\mathbf{C}^d : |z_j-\xi_j|\leqslant H/d,\ j=1,...,d\}.$$

Here,

$$c_k=\frac{1}{(2\pi i)^d}\int\cdots\int_{fr(\Pi)}\frac{f(\zeta)\,d\zeta_1\cdots d\zeta_d}{(\zeta-z)^{k+1}},\ k,\qquad 1=(1,...,1)\in(\mathbf{Z}_+)^d,$$

and $fr(\Pi)=\{z=(z_1,...,z_d)\in\mathbf{C}^d : |z_j-\xi_j|=H/d,\ \ j=1,...,d\}$. Since $M=M(f;\xi)=\max\{|f(z)| : z\in fr(\Pi_\xi)\}<\infty$, the Cauchy Inequalities imply that

$$|c_k|\leqslant M(d/H)^{|k|_1},\qquad k\in(\mathbf{Z}_+)^d.$$

If $|x-\xi|\leqslant H(e^2d)$ and if $m\in\mathbf{N}$ is large enough, then

$$\begin{aligned}|f(x)-T_m(x;\xi)| &\leqslant M\sum_{|k|_1>m}(d/H)^{|k|_1}\,(H/e^2d)^{|k|_1}\\ &=M\sum_{v=m+1}^{\infty}\exp\{-2v\}\sum_{|k|_1=v}1\\ &\leqslant M\sum_{v=m+1}^{\infty}v^d\exp\{-2v\}\leqslant M\exp\{-m\}.\end{aligned}$$

Therefore, $f \in T\{\gamma_m; K(\xi)\}$ where $K(\xi) = M(f; \xi)$ provided that $\limsup_{m\to\infty} \gamma_m < H/(e^2 d)$. Theorem 7 is proved.

Proof of Theorem 8. The implications (ii) ⇒ (iii), (ii) ⇒ (iv), (iii) ⇒ (v), and (iv) ⇒ (v) are evident. So, we should prove the validity of the implications (i) ⇒ (ii) and (v) ⇒ (i).

For some particular cases the implication (i) ⇒ (ii) was verified earlier (see [Lo1] and [Lo2]). Here we will prove the general case. We need the following lemma:

LEMMA 1. *Given* $d \in \mathbf{N}$, $\rho \in (0, \infty)$, *and* $\sigma \in (0, \infty)$, *let a finite number* q *satisfy the inequality*

$$q^\rho > e\gamma\rho\sigma \tag{22}$$

where $\gamma = \max\{1, 2^{\rho-1}\}$. *Then*

$$\begin{aligned} &\forall f \in [\rho, \sigma]_d\ \forall \varepsilon > 0\ \exists C < \infty\ \forall z^{(0)} \in \mathbf{C}^d\ \forall m \in \mathbf{N}: \\ &\max\{|f(z) - T_m(z; z^{(0)})| : |z - z^{(0)}| \leqslant m^{1/\rho}/q\} \\ &\leqslant C \exp\{\gamma(\sigma+\varepsilon)\, |z^{(0)}|^\rho\} \left(\frac{e\gamma\rho(\sigma+\varepsilon)}{q^\rho}\right)^m. \end{aligned} \tag{23}$$

Proof of Lemma 1. Let q satisfy inequality (22). It is enough to prove (23) only for small $\varepsilon > 0$. So, we can assume that

$$q^\rho > e\gamma\rho(\sigma+\varepsilon).$$

Let

$$C = \sup\{|f(z)| \exp\{-(\sigma+\varepsilon/2)\, |z|^\rho\} : z \in \mathbf{C}^d\}.$$

By the Cauchy Inequalities we have

$$\begin{aligned} &|D^k f(z^{(0)})/k!| \\ &\leqslant \min\{\min\{\max\{|f(z^{(0)}+\varsigma)|/(r_1^{k_1}\cdots r_d^{k_d}) : |\varsigma_j| = r_j;\ j = 1, ..., d\} : \\ &\qquad r_1^2 + \cdots + r_d^2 = r^2\} : r > 0\} \\ &\leqslant C \exp\{\gamma(\sigma+\varepsilon/2)\, |z^{(0)}|^\rho\} \left(\frac{e\gamma\rho(\sigma+\varepsilon/2)}{|k|_1}\right)^{|k|_1/\rho} \Bigg/ \sqrt{\prod_{j=1}^{d}\left(\frac{k_j}{|k|_1}\right)^{k_j}}. \end{aligned} \tag{24}$$

Using this estimate, we obtain

$$\max\{|f(z)-T_m(z;z^{(0)})| : |z-z^{(0)}|\leqslant m^{1/\rho}/q\}$$

$$\leqslant \max\left\{\sum_{|k|_1>m} |D^k f(z^{(0)})(z-z^{(0)})^k|/k! : |z-z^{(0)}|\leqslant m^{1/\rho}/q\right\}$$

$$\leqslant C\exp\{\gamma(\sigma+\varepsilon/2)\,|z^{(0)}|^\rho\} \sum_{|k|_1>m}\left(\frac{e\gamma\rho(\sigma+\varepsilon/2)}{q^\rho\,|k|_1}\right)^{|k|_1/\rho}$$

$$\times \max\{r_1^{k_1}\cdots r_d^{k_d} : r_1^2+\cdots+r_d^2=(m^{1/\rho}/q)^2\}\bigg/\sqrt{\prod_{j=1}^d\left(\frac{k_j}{|k|_1}\right)^{k_j}}$$

$$= C\exp\{\gamma(\sigma+\varepsilon/2)\,|z^{(0)}|^\rho\} \sum_{\nu>m}\left(\frac{e\gamma\rho(\sigma+\varepsilon/2)m}{q^\rho\nu}\right)^{\nu/\rho} \sum_{|k|_1=\nu} 1$$

$$= C\exp\{\gamma(\sigma+\varepsilon/2)\,|z^{(0)}|^\rho\} \sum_{\nu>m}\left(\frac{e\gamma\rho(\sigma+\varepsilon/2)m}{q^\rho\nu}\right)^{\nu/\rho} \nu^d$$

$$= C\exp\{\gamma(\sigma+\varepsilon/2)\,|z^{(0)}|^\rho\} \sum_{\nu>m}\left(\frac{e\gamma\rho(\sigma+\varepsilon)m}{q^\rho\nu}\right)^{\nu/\rho}$$

$$= C\exp\{\gamma(\sigma+\varepsilon/2)\,|z^{(0)}|^\rho\} \sum_{\nu>m}\left(\frac{e\gamma\rho(\sigma+\varepsilon)}{q^\rho}\right)^{\nu/\rho}$$

$$= C\exp\{\gamma(\sigma+\varepsilon)\,|z^{(0)}|^\rho\} \left(\frac{e\gamma\rho(\sigma+\varepsilon)}{q^\rho}\right)^{m}.$$

Lemma 1 is proved.

Now we can easily prove the validity of the implication (i) ⇒ (ii). Let $f\in[\rho;\sigma]_d$ for some $\sigma\in(0,\infty)$. Let $\varepsilon=1$, and let $p=q^{-1}$ where

$$q=e^2\gamma\rho(\sigma+1).$$

By Lemma 1 the estimate

$$\max\{|f(x)-T_m(x;\xi)| : |x-\xi|\leqslant m^{1/\rho}/q\}$$
$$\leqslant C\exp\{\gamma(\sigma+1)\,|\xi|^\rho-m\} = C\exp\{\gamma(\sigma+1)\,|\xi|^\rho\}\,\exp\{-m\}$$

holds for every $\xi\in\mathbf{R}^d$ and every natural m. The implication (i) ⇒ (ii) is true.

Let (v) be fulfilled, and let

$$0<\lambda<\sqrt[3]{e}-1. \tag{25}$$

Put $n_\nu = 3^{\nu/\rho} q$, $\nu \in \mathbf{N}$. The series

$$P_{3^\nu}(x) + \sum_{j=0}^{\infty} \{P_{3^{\nu+j+1}}(x) - P_{3^{\nu+j}}(x)\} = P_{3^\nu}(x) + \sum_{j=0}^{\infty} Q_j(x)$$

converges to $f(x)$ uniformly on the ball $\{x \in \mathbf{R}^d : |x| \leqslant n_\nu\}$. By the S. Bernstein Inequality the estimate

$$|Q_j(z)| \leqslant 2C \exp\{-3^{\nu+j}\}(1+\lambda)^{3^{\nu+j}}, \qquad j \in \mathbf{Z}_+, \tag{26}$$

is valid at every point z of the product of d ellipsis with foci and $\pm n_\nu$ and the sum of semiaxes equal to $(1+\lambda) n_\nu$. By (25) the series (26) converges uniformly on this product. Its sum is an analytic extension of f. Since ν is arbitrary, the function f is entire.

Let us prove that $f \in [\rho; \infty)_d$. The absolute value of the difference between $|f|$ and $|P_{3^\nu}|$ is bounded by some positive constant independent of ν on the polydisk $\{z \in \mathbf{C}^d : |z_j| \leqslant \lambda n_\nu,\ j = 1, ..., d\}$. To estimate

$$\max\{|P_{3^\nu}(z)| : |z_j| \leqslant \lambda n_\nu, j = 1, ..., d\},$$

we introduce two sequences:

$$M_\mu = \max\{|P_{3^\mu}(x)| : |x_j| \leqslant n_\mu, j = 1, ..., d\}, \qquad \mu = 0, 1, ...;$$
$$m_\mu = \max\{|P_{3^\mu}(x)| : |x_j| \leqslant n_{\mu-1}, j = 1, ..., d\}, \qquad \mu = 1, 2, ...$$

For all natural μ we have

$$m_\mu \leqslant M_{\mu-1} + 2C \exp\{-3^{\mu-1}\}.$$

Besides,

$$M_\mu \leqslant m_\mu (1 + 2h + 2\sqrt{h + h^2})^{3^\mu d}$$

where $2h = 3^{1/\rho} - 1$ (see [N, Ch. IV]). Both of these inequalities imply that

$$\begin{aligned} M_\mu &\leqslant m_\mu \kappa^{3^\mu d} \leqslant (M_{\mu-1} + 2C \exp\{-3^{\mu-1}\})\, \kappa^{3^\mu d} \leqslant M_{\mu-1} \eta^{3^\mu d} \\ &\leqslant M_{\mu-2} \eta^{3^\mu d + 3^{\mu-1} d} \leqslant \cdots \leqslant M_0 \eta^{(3^\mu + \cdots + 1) d} \leqslant M_0 \omega^{3^\mu d}. \end{aligned}$$

Here κ, η, and ω are some constants that do not depend on d. Therefore, we have the estimate

$$\begin{aligned} \max\{|P_{3^\nu}(z)| : |z_j| \leqslant \lambda n_\nu,\ j = 1, ..., d\} &\leqslant M_0 \omega^{O((\lambda n_\nu)^\rho)} \\ &= \exp\{O((\lambda n_\nu)^\rho)\}, \qquad \nu \to \infty. \end{aligned}$$

Consequently,

$$\max\{|f(z)| : |z_j| \leqslant \lambda n_\nu, j = 1, ..., d\} \leqslant \exp\{O((\lambda n_\nu)^\rho)\}, \qquad \nu \to \infty.$$

Since $M_f(r)=\max\{|f(z)| : |z_j|\leqslant r, j=1, ..., d\}$ is a monotone function and $n_\nu/n_{\nu-1}=3^{1/\rho}$, we have $f\in[\rho, \infty)_d$. Theorem 8 is proved.

APPLICATIONS (PROOFS OF THEOREMS 9 AND 10)

Proof of Theorem 9. Assume that $\sigma=1$—we can always reduce the general case to this particular one using the dilatation $z\mapsto z/\sigma$. Let $f\in[1; 1]_d$. According to estimate (24), for any $k\in(\mathbf{Z}_+)^d\backslash\{(0, ..., 0)\}$ we have

$$|D^k f(0)|/k!\leqslant C\left(\frac{e(1+\varepsilon/2)}{|k|_1}\right)^{|k|_1}\Bigg/\sqrt{\prod_{j=1}^{d}\left(\frac{k_j}{|k|_1}\right)^{k_j}}$$

$$<C\left(\frac{2e}{|k|_1}\right)^{|k|_1}\Bigg/\sqrt{\prod_{j=1}^{d}\left(\frac{k_j}{|k|_1}\right)^{k_j}}. \tag{24}$$

Besides, by Lemma 1 for every $m\in\mathbf{N}$

$$\max\{|f(z)-T_m(z;0)| : |z|\leqslant m/q\}\leqslant C\left(\frac{e(1+\varepsilon)}{q}\right)^m<C\left(\frac{2e}{q}\right)^m<C\left(\frac{2}{3}\right)^m.$$

Here, $C=\sup\{|f(z)|\exp\{-(1+\varepsilon/2)\,|z|\} : z\in\mathbf{C}^d\}$, $0<\varepsilon<1$, and $q=[3e]+1$. Let

$$\varphi_m(z)=T_m(z;0)\left\{\frac{\sin(2q\sqrt{z_1^2+\cdots+z_d^2}/m)}{2q\sqrt{z_1^2+\cdots+z_d^2}/m}\right\}^{qm},\qquad m\in\mathbf{N}.$$

It turns out that the sequence $\{\varphi_m\}$ has the following properties:

(i) $\{\varphi_m\}_{m=1}^{\infty}\subset[1; 2q^2]_d$;

(ii) $\{\varphi_m\}$ converges to f, and this convergence is uniform on each compact subset of $\mathbf{C}^d$;

(iii) $|x|\leqslant\sqrt{m}/(2q\sqrt{q})\Rightarrow|\varphi_m(x)|\geqslant 5\,|f(x)|/6+o(1),\qquad m\to\infty$;

(iv) $|x|\leqslant m/q\Rightarrow|\varphi_m(x)|\leqslant|f(x)|+o(1),\qquad m\to\infty$;

(v) $|x|\geqslant m/q\Rightarrow|\varphi_m(x)|=o(1),\qquad m\to\infty$.

Before verifying these properties, notice that all functions φ_m are bounded on the real hyperplane $\mathbf{R}^d$. We will presently have the occasion to use this boundedness.

(i) and (ii) are evident.

(iii) Let $|x| \leqslant \sqrt{m}/(2q\sqrt{q})$. Since

$$1 - \left\{\frac{\sin(2q\,|x|/m)}{2q\,|x|/m}\right\}^{qm} \leqslant qm\left(1 - \frac{\sin(2q\,|x|/m)}{2q\,|x|/m}\right)$$
$$\leqslant \frac{1}{3!}\, qm(2q\,|x|/m)^2 \leqslant 2q^3\,|x|^2/(3m) \leqslant 1/6,$$

the estimate

$$|\varphi_m(x)| \geqslant |f(x)| - |f(x) - T_m(x;0)| - |T_m(x;0)| \left\{1 - \left\{\frac{\sin(2q\,|x|/m)}{2q\,|x|/m}\right\}^{qm}\right\}$$
$$\geqslant |f(x)| - |f(x) - T_m(x;0)| \left\{2 - \left\{\frac{\sin(2q\,|x|/m)}{2q\,|x|/m}\right\}^{qm}\right\}$$
$$- |f(x)| \left\{1 - \left\{\frac{\sin(2q\,|x|/m)}{2q\,|x|/m}\right\}^{qm}\right\}$$
$$\geqslant 5\,|f(x)|/6 + o(1), \qquad m \to \infty$$

is valid. So, (iii) is proved.

(iv) Let $|x| \leqslant m/q$. By Lemma 1

$$|\varphi_m(x)| \leqslant |T_m(x;0)| \leqslant |f(x)| + |f(x) - T_m(x;0)| = |f(x)| + o(1), \quad m \to \infty.$$

(v) (27) implies the inequality

$$|T_m(x;0)| \leqslant C \sum_{|k|_1 \leqslant m} \left(\frac{2e}{|k|_1}\right)^{|k|_1} |x_1|^{k_1} \cdots |x_d|^{k_d} \Big/ \sqrt{\prod_{j=1}^{d} \left(\frac{k_j}{|k|_1}\right)^{k_j}}.$$

Therefore,

$$\max\{|T_m(x;0)| : |x| = r\} \leqslant C \sum_{|k|_1 \leqslant m} \left(\frac{2er}{|k|_1}\right)^{|k|_1}.$$

Let $r = |x| \geqslant m/q$. Then

$$\max\{|\varphi_m(x)| : |x| = r\} \leqslant C \left|\frac{\sin(2qr/m)}{2qr/m}\right|^{qm} \sum_{|k|_1 \leqslant m} \left(\frac{2er}{|k|_1}\right)^{|k|_1}$$
$$\leqslant C \left\{\frac{m}{2qr}\right\}^{qm} \sum_{v \leqslant m} \left(\frac{2er}{v}\right)^{v} v^d$$
$$\leqslant C \left\{\frac{1}{2}\right\}^{qm} \sum_{v \leqslant m} \left(\frac{3em}{vq}\right)^{v}$$

$$\leqslant C\left\{\frac{1}{2}\right\}^{qm} \sum_{\nu \leqslant m} \frac{(4m/q)^\nu}{\nu!}$$
$$\leqslant C\{e/2^q\}^m = o(1), \qquad m \to \infty.$$

(v) is proved.

Now, the proof is easy. First assume that E is an ε-net satisfying the inequality

$$2q^2\varepsilon < 1.$$

Under this assumption the statement of Theorem 9 can be proved as follows: Let

$$L = \sup\{|f(\xi)| : \xi \in E\}. \tag{28}$$

Only the case where $L < \infty$ is of interest for us. By (iv) and (v)

$$\sup\{|\varphi_m(\xi)| : \xi \in E\} \leqslant L + o(1), \qquad m \to \infty.$$

Let

$$\Phi_m = \sup\{|\varphi_m(x)| : x \in \mathbf{R}^d\}, \qquad m \in \mathbf{N}.$$

Without loss of generality, we can assume that there exists such a point $x^{(m)} \in \mathbf{R}^d$ that $|\varphi_m(x^{(m)})| = \Phi_m$. By the definition of ε-net there exists a point $\xi^{(m)} \in E$ satisfying the inequality $|x^{(m)} - \xi^{(m)}| < \varepsilon$. We have

$$|\varphi_m(x^{(m)}) - \varphi_m(\xi^{(m)})| \leqslant |x^{(m)} - \xi^{(m)}| \sup\{|\text{grad } \varphi_m(y)| : y \in \mathbf{R}^d\}$$
$$\leqslant \varepsilon \sup\{|\text{grad } \varphi_m(y)| : y \in \mathbf{R}^d\}.$$

By S. Bernstein's estimate of the derivative of entire function of exponential type that is bounded on the real hyperplane the inequality

$$\sup\{|\text{grad } \varphi_m(y)| : y \in \mathbf{R}^d\} \leqslant 2q^2\Phi_m$$

holds. Two previous inequalities show that

$$|\varphi_m(x^{(m)}) - \varphi_m(\xi^{(m)})| \leqslant 2q^2\varepsilon\Phi_m$$

or

$$\Phi_m - (L + o(1)) \leqslant 2q^2\varepsilon\Phi_m.$$

It means that the estimate

$$\Phi_m \leqslant (L + o(1))/(1 - 2q^2\varepsilon), \qquad m \to \infty,$$

is valid. Recalling (ii), we see that

$$\sup\{|f(x)| : x \in \mathbf{R}^d\} \leqslant L/(1-2q^2\varepsilon).$$

Using more sophisticated technique than S. Bernstein's estimate, Beurling [Beu2] proved that the inequality

$$\sup\{|f(x)| : x \in \mathbf{R}^d\} \leqslant CL$$

is valid for every entire function of exponential type $2q^2$ provided that *it is a priori bounded on the real hyperplane and* $2q^2\varepsilon < \pi$. Here, C does not depend on f, and L is defined by (28). By Beurling's result f is bounded on $\mathbf{R}^d$ upon this, weaker, condition.

We just proved that f is bounded on the real hyperplane. To finish the proof, we should only repeat the estimate used for φ_m for f itself substituting 1 for $2q^2$ and recall that $2q^2 < 200$. Theorem 9 is proved.

Proof of Theorem 10. Let E be such a 1-net in $\mathbf{R}^d$ that any affine subspace of dimension less than d contains at most of d its points and

$$\inf\{|\xi-\eta| : \xi,\ \eta \in E,\ \xi \neq \eta\} > 0.$$

Let also $g_j(\zeta)$, $j=1, ..., d$, be such entire functions of exponential type 1 in $\mathbf{C}$ that $|g_j(\tau)| \geqslant 1$ on the real line,

$$\forall j \in \{1, ..., d\}\ \forall m \in \mathbf{N} : \lim_{\tau \to \pm\infty} \tau^m/g_j(\tau) = 0,$$

and

$$\int_{-\infty}^{\infty} \frac{\log^+ |g_j(\tau)|\, d\tau}{1+\tau^2} < \infty.$$

Define a weight-function $\varphi(\xi)$, $\xi = (\xi_1, ..., \xi_d)$, as $|\prod_{j=1}^d g_j(\xi_j)|$ on E and set $\varphi(x) = \exp\{|x|\}$ outside E.

Let us check the fulfillment of properties (i)–(iv) of φ.

(i) and (ii) are evident.

(iii) Without loss of generality, we can assume that A is the subspace $\{x = (x_1, ..., x_d) \in \mathbf{R}^d\!: x_{l+1} = \cdots = x_d = 0\}$, $1 \leqslant l < d$. For a while, we will denote vectors of $\mathbf{R}^l$ by x and the restriction $\varphi|_{\mathbf{R}^l}$ by the same letter φ. Assume that algebraic polynomials are not dense in the space $C^0_\varphi(\mathbf{R}^l)$. The dual space of $C^0_\varphi(\mathbf{R}^l)$ consists of all such measures μ that

$$\int_{\mathbf{R}^l} \varphi(x)\, d\,|\mu|(x) < \infty. \tag{29}$$

If polynomials are not dense in $C^0_\varphi(\mathbf{R}^i)$, then the space of Fourier transforms of measures satisfying inequality (29) is not Δ-quasianalytic. Really, let μ be an orthogonal measure to all polynomials. Define

$$F(t)=\int_{\mathbf{R}^l} \exp\{i\langle t, x\rangle\}\, d\mu(x).$$

where $\langle t, x\rangle=\sum_{j=1}^{l} t_j x_j$ is a scalar product in $\mathbf{R}^l$. It is evident that $F\in C^\infty(\mathbf{R}^l)$ and that for every $k\in(\mathbf{Z}_+)^l$

$$D^k F(0)=0.$$

Since $\varphi(x)=\exp\{|x|\}$ everywhere except at most at d points, F is indeed a function holomorphic in some layer symmetric with respect to $\mathbf{R}^l$. Therefore, $F(t)\equiv 0$, $\mu=0$, and algebraic polynomials are dense in $C^0_\varphi(\mathbf{R}^l)$.

(iv) Denote by $\mathfrak{R}$ a set of all algebraic polynomials P such that their norms in $C^0_\varphi(\mathbf{R}^d)$ satisfy the inequality

$$\left\| P(x)\Big/\prod_{j=1}^{d}(x_j-i)\right\|\leqslant 1.$$

Let

$$M(z)=\sup\{|P(z)|: P\in\mathfrak{R}\},\qquad z\in\mathbf{C}^d.$$

We will prove that if algebraic polynomials are dense in the space $C^0_\varphi(\mathbf{R}^d)$, then $M(z)=\infty$ for all $z=(z_1, ..., z_d)$ such that $\operatorname{Im} z_1\cdots\operatorname{Im} z_d\neq 0$. This statement seems to be well-known (for the case where $d=1$ it is the well-known Mergelian criterion of density [Me]), but we suspect that for the case where $d>1$ it exists only as mathematical folk-lore. For the comfort of the readers, we provide its proof here.

Assume that algebraic polynomials are dense in $C^0_\varphi(\mathbf{R}^d)$. Using the description of dual space and Stiltjes Inversion Formula, one can easily prove that the system $\{1/\prod_{j=1}^{d}(x_j-z_j)\colon \operatorname{Im} z_1\cdots\operatorname{Im} z_d\neq 0\}$ is complete in $C^0_\varphi(\mathbf{R}^d)$. Let $\varepsilon>0$, and let $z=(z_1, ..., z_d)$, $\operatorname{Im} z_1\cdots\operatorname{Im} z_d\neq 0$, be an arbitrary point of $\mathbf{C}^d$. There exists a polynomial P satisfying the inequality

$$\left\| 1\Big/\prod_{j=1}^{d}(x_j-z_j)-P(x)\right\|<\varepsilon.$$

For some finite value $C_z>0$ that does not depend on ε the polynomial

$$Q(x)=\left(1-P(x)\prod_{j=1}^{d}(x_j-z_j)\right)\Big/(C_z\varepsilon)\in\mathfrak{R}.$$

On the other hand, $|Q(z)|=1/(C_z\varepsilon)$. Since $\varepsilon>0$ is arbitrary, $M(z)=\infty$.

By the Beurling–Malliavin Theorem on multiplicator [BM] (see also [K]) for every $\eta > 0$ there exist nontrivial entire functions $h_j(\varsigma) \in [1; \eta]_1$, $j = 1, ..., d$, such that

$$\sup\{ |(\tau - i)\, g_j(\tau)\, h_j(\tau)| : \tau \in \mathbf{R}\} \leqslant 1.$$

Therefore, the inequality

$$\sup \left\{ \left| P(\xi) \prod_{j=1}^{d} h_j(\xi_j) \right| : \xi = (\xi_1, ..., \xi_d) \in E \right\} \leqslant 1$$

holds for every polynomial $P \in \mathfrak{R}$. Certainly, we can choose η so that

$$200\eta \sqrt{d} < \pi.$$

Since the exponential type of entire function $P(z) \prod_{j=1}^{d} h_j(z_j)$ does not exceed $\eta \sqrt{d}$, Theorem 9 shows that

$$\sup \left\{ \left| P(x) \prod_{j=1}^{d} h_j(x_j) \right| : x = (x_1, ..., x_d) \in \mathbf{R}^d \right\} \leqslant C$$

where C does not depend on P. Using the Phragmen–Lindelöf Principle, we see that the estimate

$$\left| P(z) \prod_{j=1}^{d} h_j(z_j) \right| \leqslant C \exp\{\eta(|\mathrm{Im}\, z_1| + \cdots + \mathrm{Im}\, z_d|)\}$$

is valid for every $z \in \mathbf{C}^d$ and every $P \in \mathfrak{R}$. Let $z^{(0)} \in \mathbf{C}^d$, $\mathrm{Im}\, z_1^{(0)} \cdots \mathrm{Im}\, z_d^{(0)} \neq 0$, be such a point that $\prod_{j=1}^{d} h_j(z_j^{(0)}) \neq 0$. Then

$$M(z^{(0)}) \leqslant C \exp\{\eta(|\mathrm{Im}\, z_1^{(0)}| + \cdots + |\mathrm{Im}\, z_d^{(0)}|)\} \Big/ \prod_{j=1}^{d} |h_j(z_j^{(0)})| < \infty.$$

Therefore, algebraic polynomials are not dense in $C_\varphi^0(\mathbf{R}^d)$. Theorem 10 is proved.

Acknowledgment

The authors are grateful to the referees and the editor for their useful remarks.

References

[A] S. Agmon, Functions of exponential type in an angle and singularities of Taylor series, *Trans. Amer. Math. Soc.* **70** (1951), 492–508.

[Ak] N. I. Akhiezer, Interpolation of entire transcendental functions of finite degree, *Dokl. Akad. Nauk SSSR* **65** (1949), 781–784. [in Russian]

[Ber] S. N. Bernstein, Extension of properties of trigonometric polynomials on entire functions of finite degree, *Izv. Akad. Nauk SSSR* **12** (1945), 421–444. [in Russian]

[Beu1] A. Beurling, "On Quasianalyticity and General Distributions," mimeographed lecture notes, Stanford University, 1961.

[Beu2] A. Beurling, On balayage of Fourier–Stiltjes transforms, *in* "The Collected Works of Arne Beurling, Vol. 2, Harmonic Analysis," Birkhauser, Boston, 1989.

[BM] A. Beurling and P. Malliavin, On Fourier transforms of measures with compact support, *Acta Math.* **107** (1962), 79–93.

[Bo] R. P. Boas, Asymptotic properties of functions of exponential type, *Duke Math. J.* **20** (1953), 433–448.

[Br] A. Yu. Brudnyi, Local theory of polynomial approximation, *Theory Functions Funct. Anal. Appl.* **36** (1981), 3–12. [in Russian]

[BS] R. P. Boas and A. C. Schaeffer, A theorem of Cartwright, *Duke Math. J.* **9** (1942), 879–883.

[C] M. L. Cartwright, On certain integral functions of order 1, *Quart. J. Math.* **7** (1936), 46–55.

[DL] H. E. Davidova and V. N. Logvinenko, On entire functions of exponential type bounded on the real line, in press.

[K] P. Koosis, "The Logarithmic Integral," Vols. 1, 2, Cambridge Univ. Press, Cambridge, 1988, 1992.

[L] P. Lelong, *C.R.* **232** (1951), 1178. [in French]

[Le1] B. Ya. Levin, On entire functions of finite degree bounded on a sequence of points, *Dokl. Akad. Nauk SSSR* **65** (1949), 265–268. [in Russian]

[Le2] B. Ya. Levin, Generalization of Cartwright's theorem on entire function bounded on some sequence of real points, *Izv. Akad. Nauk SSSR* **21** (1957), 549–556. [in Russian]

[Le3] B. Ya. Levin, Majorants of some classes of subharmonic functions, preprint #18-84, FTINT Ukrainian AN, 1984, 52 pp. [in Russian]

[LL] B. Ya. Levin and V. N. Logvinenko, On classes of subharmonic functions in $\mathbf{R}^n$ bounded on some subsets, *in* "Coll. Dedicated to Centenary of V. I. Smirnov," pp. 157–175, Soviet Math. Institute (Leningrad Branch), 1989. [in Russian]

[LLS] B. Ya. Levin, V. N. Logvinenko, and M. L. Sodin, Subharmonic Functions of Finite Degree Bounded on Subsets of "Real Hyperplane," *Adv. Soviet Math.* **11** (1992), 187–197.

[Lo1] V. N. Logvinenko, On a multidimensional generalization of a theorem of M. L. Cartwright, *Soviet Math. Dokl.* **5** (1974), 1617–1620.

[Lo2] V. N. Logvinenko, Boundedness and slow growth of entire functions of exponential type along the real hyperplane, *Sibirsk. Mat. Zh.* **29** (1988), 126–138. [in Russian]

[Lo3] V. N. Logvinenko, Some conditions of positive definiteness of entire functions of exponential type on $\mathbf{R}^n$, *Theory Functions Funct. Anal. Appl.* **56** (1991), 88–94. [in Russian]

[Lo4] V. N. Logvinenko, Theorem on separate analyticity and edge-of-the-wedge theorem, *Theory Functions Funct. Anal. Appl.* **57** (1993), 3–14. [in Russian]

[LS] V. N. Logvinenko and L. E. Sokolova, On support theorem for Radon transform, in press.

[M] P. Malliavin, Sur le croissance radialle d'une fonction meromorphe, *Illinois J. Math.* (1957).

[Ma] S. Mandelbrojt, "Quasianalytic Classes of Functions," ONTI, Leningrad–Moscow, 1937. [in Russian]

[Me] S. N. Mergelian, Weighted approximation by polynomials, *Amer. Math. Soc. Transl. Ser. 2* **10** (1958), 59–106.

[MR] V. I. Matsaev and L. I. Ronkin, Quasianalytic classes of functions of several variables, *Proc. Math. Department Kharkov Univ. Kharkov Math. Soc. Ser. 4* **27** (1961), 75–84. [in Russian]

[N] I. P. Natanson, "Constructive Theory of Functions," GITTL, Moscow–Leningrad, 1949. [in Russian]

[S] A. C. Schaeffer, Entire functions and trigonometric polynomials, *Duke Math. J.* **20** (1953), 77–78.